# F-invariant in cluster algebras

- $V$: a vector space over $\mathbb{C}$ with $\dim V = m$.
- $f: V \longrightarrow V$ linear map.
- $\operatorname{rank}(f) \in \mathbb{Z}_{\geq 0}$ is an invariant of $f$ and each choice of a basis $(\alpha_1, \cdots, \alpha_m)$ of $V$ gives a way to calculate $\operatorname{rank}(f)$ via

$$f(\alpha_1, \cdots, \alpha_m) = (\alpha_1, \cdots, \alpha_m) A \quad \text{and} \quad \operatorname{rank}(f) = \operatorname{rank}(A).$$

- Some good choice of basis $(\alpha_1, \cdots, \alpha_m)$ can make the calculation of $\operatorname{rank}(f)$ easier, for example, in the case that $A$ is a Jordan matrix.

Today: Want to introduce F-invariant $(u \| u')_F \in \mathbb{Z}_{\geq 0}$ in cluster algebras, which behaves like $\dim \operatorname{Ext}^1(M, M')$ in a 2-CY triangulated category $\mathcal{C}$.

| Vector space | cluster alg $\subseteq \mathbb{Q}(z_1, \cdots, z_m)$ | 2-CY $\Delta$-cat $\mathcal{C}$. |
|---|---|---|
| basis $(\alpha_1, \cdots, \alpha_m)$ | cluster $X = (x_1, \cdots, x_m)$ | cluster tilting obj. $T = \bigoplus_{i=1}^{m} T_i$ |
| $\Big\}$ calculate $\downarrow$ | $\Big\}$ calculate $\downarrow$ | calculate $\Big\}$ via $\frac{\mathcal{C}}{T[1]} \simeq \operatorname{mod} \operatorname{End}(T)^{op}$ $\downarrow$ |
| $\operatorname{rank}(f)$ | $(u \| u')_F$ | $\dim \operatorname{Ext}^1(M, M')$ |

The calculation is independent of the choice of basis / cluster / cluster tilting obj.

plan: §1. Cluster algebras.

§2. F-invariant and results.

# F-invariant in cluster algebras

- Fix $m, n \in \mathbb{Z}$ with $m \geq n > 0$.
- $\tilde{B} = \begin{bmatrix} B_{n\times n} \\ P \end{bmatrix}$ : $m \times n$ integer matrix $\quad\quad \Lambda$: $m \times m$ integer matrix
- $\mathbb{F} = \mathbb{Q}(z_1, \cdots, z_m)$ ($\supseteq \mathcal{A}$ cluster algebra)

Def. Call $(\tilde{B}, \Lambda)$ a compatible pair, if $\tilde{B}^T \Lambda = (S \,|\, 0)_{n \times m}$ for some $S = \mathrm{diag}(s_1, \cdots, s_n)$ with $s_i \in \mathbb{Z}_{>0}$, $\forall i$.

Rmk: In this case, $\tilde{B} = \begin{bmatrix} B \\ P \end{bmatrix}$ has the full rank $n$ and $\tilde{B}^T \Lambda \tilde{B} = SB$ is skew-symmetric. Thus $B$ is skew-symmetrizable.

Def. Call $t_0 = (X, \tilde{B}, \Lambda)$ a seed in $\mathbb{F} = \mathbb{Q}(z_1, \cdots, z_m)$, if

① $X = (x_1, \cdots, x_m)$ is a free generating set of $\mathbb{F}$.

② $(\tilde{B}, \Lambda)$ is a compatible pair.

Call $X$ a cluster, $x_1, \cdots, x_m$ cluster variables.

- We have mutations $\mu_1, \cdots, \mu_n$ to produce new seeds

$$\mu_k : (X, \tilde{B}, \Lambda) \longmapsto (X', \tilde{B}', \Lambda')$$

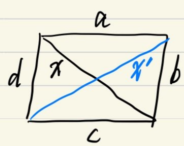
where $X' = (x_1, \cdots, x_k', \cdots, x_m)$ and the new cluster variable $x_k'$ is given by the $k$-th column of $\tilde{B} = (b_{ij})$:

$$x_k' = x_k^{-1} \Big( \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \Big) \rightsquigarrow \text{mutation relation}$$

Rmk. ① $\mu_k^2 = \mathrm{id}$ and $\tilde{B}^T \Lambda = (S \,|\, 0)_{n \times m} = \tilde{B}'^T \Lambda'$ $\rightsquigarrow$ mutation invariant

② For the case of $m = n = 2$, $\mu_k(\tilde{B}, \Lambda) = (-\tilde{B}, -\Lambda)$.

③ Mutation relation is a generalization of Ptolemy relation: $x' = x^{-1}(ac + bd)$.



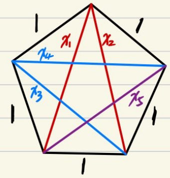
- Fix an initial seed $t_0 = (X, \tilde{B}, \Lambda)$. Denote by

$$\Delta = \{ t = \vec{\mu}(t_0) \mid \vec{\mu} \text{ any seq. of mutations} \}$$

Def. cluster alg. $\mathcal{A} = \mathbb{Z}[\text{all cluster variables in } \Delta] \subseteq \mathbb{F} = \mathbb{Q}(z_1, \cdots, z_m)$.

Example. $\tilde{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \Lambda$, $\tilde{B}^T \Lambda = (S|0) = I_2$, $X = (x_1, x_2)$

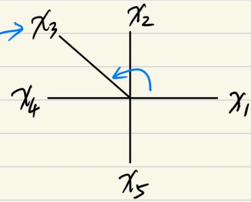
$$\underset{x_1,\, x_2}{t_0 = (X, \tilde{B}, \Lambda)} \xrightarrow{\mu_1} \underset{x_3}{t_1} \xrightarrow{\mu_2} \underset{x_4}{t_2} \xrightarrow{\mu_1} \underset{x_5}{t_3} \longrightarrow \cdots$$




Denote by $\hat{y}_1 = X^{\tilde{B}e_1} = x_2^{-1}$, $\hat{y}_2 = X^{\tilde{B}e_2} = x_1$

Mutation relation: $x_{k+2} = \dfrac{x_{k+1} + 1}{x_k} \Longrightarrow$

$x_3 = \dfrac{x_2+1}{x_1} \overset{\text{rewrite}}{=\!=\!=} x_1^{-1} x_2 \cdot (1 + \hat{y}_1) = X^{g} \cdot F(\hat{Y})$, taking g-vector




$x_4 = \dfrac{x_1 + x_2 + 1}{x_1 x_2} = x_1^{-1} \cdot (1 + \hat{y}_1 + \hat{y}_1 \hat{y}_2)$,

$x_5 = \dfrac{x_1 + 1}{x_2} = x_2^{-1} \cdot (1 + \hat{y}_2)$, $x_6 = x_1$, $x_7 = x_2$.

$\Rightarrow \mathcal{A} = \mathbb{Z}[x_1, x_2, \dfrac{x_2+1}{x_1}, \dfrac{x_1+x_2+1}{x_1 x_2}, \dfrac{x_1+1}{x_2}] \subseteq \mathbb{Q}(x_1, x_2)$.

clusters $\{x_i, x_{i+1}\}_{i=1}^{5} \rightsquigarrow$ chambers in  .

Def. A cluster monomial of $\mathcal{A}$ is a monomial in cluster variables from the same cluster (chamber), e.g., $x_3 x_4$ ✓, $x_3 x_5$ ✗.

Thm [FZ, GHKK] Let $u$ be a cluster monomial and $t = (X_t, \tilde{B}_t, \Lambda_t)$ a seed of $\mathcal{A}$. Then

(1). The expansion of $u$ w.r.t. $X_t$ is a Laurent polynomial.

(2). Set $\hat{y}_{k;t} = X_t^{\tilde{B}_t e_k}$. The expansion above can be uniquely written as $u = X_t^{g_t} \cdot F_t(\hat{Y}_t)$, where $\underbrace{g_t}_{\text{g-vector}} \in \mathbb{Z}^m$ and $\underbrace{F_t}_{\text{F-polynomial}} \in \mathbb{Z}[y_1, \cdots, y_n]$ with $y_i \nmid F_t$, $\forall i$.

(3). $F_t$ has positive coefficients and constant term 1.

§2. Given a polynomial $F=\sum_{v\in\mathbb{N}^n} c_v Y^v \in \mathbb{Z}[y_1,\cdots,y_n]$ and a vector $h\in\mathbb{Z}^n$, denote by $F[h]:=\max\{v^T h \mid c_v\neq 0\}\in\mathbb{Z}$.

Rmk: If $F$ has constant term $1$, then $F[h]\geq 0$, $\forall h\in\mathbb{Z}^n$.

Def. Let $u=X_t^{g_t}\cdot F_t(\hat{Y}_t)$ and $u'=X_t^{g_t'}\cdot F_t'(\hat{Y}_t)$ be two cluster monomials of $\mathcal{A}$ written their expansions in any seed $t=(X_t,\tilde{B}_t,\Lambda_t)$. Define an integer $\langle u,u'\rangle_t = g_t^T\Lambda_t g_t' + F_t[\underbrace{(S/0)g_t'}_{\in\mathbb{Z}^n}]$.

Thm. The integer $\langle u,u'\rangle_t$ only depends on $u$ and $u'$, not on the choice of $t$.

pf. Consider the g-vectors of $u'$ w.r.t. different seeds $\rightsquigarrow$

$$\{g_w'\in\mathbb{Z}^m \mid w\in\Delta\} \xrightarrow{\Lambda} \{\Lambda_w g_w'\in\mathbb{Z}^m \mid w\in\Delta\}.$$

Denote by $\mathbb{Q}_{sf}(x_1,\cdots,x_m)=\{\frac{P}{Q} \mid 0\neq P,Q\in\mathbb{Z}_{\geq 0}[x_1,\cdots,x_m]\}$.

Clearly, $\mathbb{Q}_{sf}(x_1,\cdots,x_m)=\mathbb{Q}_{sf}(X_w)$ for any seed $w\in\Delta$.

claim: There exists a unique semifield homomorphism

$$\beta_{u'}:(\mathbb{Q}_{sf}(x_1,\cdots,x_m),\ \cdot\ ,+)\longrightarrow(\mathbb{Z},+,\max\{\ \})$$

s.t. $\beta_{u'}(X_w)=(\Lambda_w g_w')^T\in\mathbb{Z}^m_{\text{row}}$ for any seed $w\in\Delta$.

So each choice of a cluster $X_w$ gives a way to calculate $\beta_{u'}(u)$ by writting $u$ as $u=X_w^{g_w}\cdot F_w(\hat{Y}_w)=X_w^{g_w}\cdot F_w(X_w^{\tilde{B}_w})$.

Thus $\beta_{u'}(u)=g_w^T\underline{\Lambda_w g_w'}+F_w[\tilde{B}_w^T\underline{\Lambda_w g_w'}]$

$$=g_w^T\Lambda_w g_w'+F_w[(S/0)g_w']=\langle u,u'\rangle_w.$$

Notice that the value of $\beta_{u'}(u)$ does not depend on the choice of $w$.

Def. The F-invariant between two cluster monomials $u$ and $u'$ is defined by

$$(u\|u')_F = \langle u, u'\rangle_t + \langle u', u\rangle_t$$
$$= g_t^T \Lambda_t g_t' + F_t[(S|0)g_t'] + g_t'^T \Lambda_t g_t + F_t'[(S|0)g_t]$$
$$= F_t[(S|0)g_t'] + F_t'[(S|0)g_t]$$

Rmk. (1) Since $F_t$ and $F_t'$ have constant term 1, $(u\|u')_F \geq 0$.

(2). If $u$ and $u'$ are two (unfrozen) cluster variables, say $u = x_{i;t}$, $u' = x_{j;w}$.
Then by using $g_t = e_i \in \mathbb{Z}^m$ and $F_t = 1$, we have

$$(u\|u')_F = F_t[(S|0)g_t'] + F_t'[(S|0)g_t] = F_t'[(S|0)e_i] = s_i f_i',$$

where $s_i$ is the $(i,i)$-entry of $S = \mathrm{diag}(s_1, \cdots, s_n)$ and $f_i'$ is the max. exponent of $y_i$ in $F_t'$. Thus

$$(x_{i;t} \| x_{j;w})_F = s_i f_i' = s_i \underline{(x_{i;t} \| x_{j;w})_f}$$ the f-compatibility degree defined by Fu-Gyoda.

Thm. [Fu-Gyoda] $(x_{i;t} \| x_{j;w})_f = 0$ iff $x_{i;t} \cdot x_{j;w}$ is a cluster monomial.

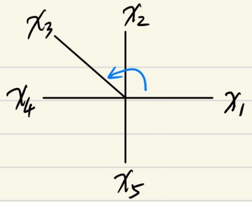
Thm. For two cluster monomials $u$ and $u'$, their product $u \cdot u'$ is still a cluster monomial iff $(u\|u')_F = 0$.

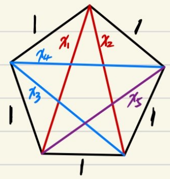
pf: "$\Longrightarrow$" Say $u \cdot u'$ is a cluster monomial in seed $t$. Then

$$F_t = 1 = F_t' \Rightarrow (u\|u')_F = F_t[\quad] + F_t'[\quad] = 0.$$

"$\Longleftarrow$" ... and use Fu-Gyoda's Thm ... .

Back to the $A_2$ example.




$x_3 = x_1^{-1} x_2 (1+\hat{y}_1)$ $\quad$ $x_5 = x_2^{-1}(1+\hat{y}_2)$

$x_4 = x_1^{-1}(1+\hat{y}_1+\hat{y}_1\hat{y}_2)$ $\quad$ $(S|0) = I_2$.

We have $(x_3 \| x_4)_F = F[g'] + F'[g] = (1+y_1)\begin{bmatrix}-1\\0\end{bmatrix} + (1+y_1+y_1y_2)\begin{bmatrix}-1\\1\end{bmatrix}$

$$= \max\{0,-1\} + \max\{0,-1,0\} = 0$$

$(x_3 \| x_5)_F = (1+y_1)\begin{bmatrix}0\\-1\end{bmatrix} + (1+y_2)\begin{bmatrix}-1\\1\end{bmatrix} = \max\{0,0\} + \max\{0,1\} = 1 > 0.$

$\Rightarrow$ $x_3x_4$ is a cluster monomial, while $x_3x_5$ is not.

Rmk. $(x_3 \| x_5)_F = 1$ $\rightsquigarrow$ the diagonals $x_3$ and $x_5$ have 1 intersection point in the interior of the 5-gon.

Rmk: F-invariant is related with

① Fomin-Zelevinsky's compatibility degree defined on almost positive roots.

② Fu-Gyoda's f-compatibility degree defined on cluster variables.

③ Derksen-Weyman-Zelevinsky's E-invariant in the additive categorification of cluster algebras, which is related with $\dim \mathrm{Ext}^1(M,N)$.

④ Kang-Kashiwara-Kim-Oh's d-invariant in the monoidal categorification of cluster algebras, which is related with of R-matrices $r_{M,N}$ and $r_{N,M}$.

$$M \otimes N \underset{r_{N,M}}{\overset{r_{M,N}}{\rightleftarrows}} N \otimes M$$